J. Fluid Mech. (2001), *vol.* 439, *pp.* 403–412. Printed in the United Kingdom

Estimating amplitude ratios in boundary layer stability theory: a comparison between two approaches

By RAMA GOVINDARAJAN AND R. NARASIMHA

Fluid Dynamics Unit, Jawaharlal Nehru Centre for Advanced Scientific Research, Jakkur, Bangalore 560 064, India
e-mail: rama@jncasr.ac.in, roddam@caos.iisc.ernet.in

(Received 20 January 2001 and in revised form 16 April 2001)

We first demonstrate that, if the contributions of higher-order mean flow are ignored, the parabolized stability equations (Bertolotti *et al.* 1992) and the 'full' non-parallel equation of Govindarajan & Narasimha (1995, hereafter GN95) are both equivalent to order R^{-1} in the local Reynolds number R to Gaster's (1974) equation for the stability of spatially developing boundary layers. It is therefore of some concern that a detailed comparison between Gaster (1974) and GN95 reveals a small difference in the computed amplitude ratios. Although this difference is not significant in practical terms in Blasius flow, it is traced here to the approximation, in Gaster's method, of neglecting the change in eigenfunction shape due to flow non-parallelism. This approximation is not justified in the critical and the wall layers, where the neglected term is respectively $O(R^{-2/3})$ and $O(R^{-1})$ compared to the largest term. The excellent agreement of GN95 with exact numerical simulations, on the other hand, suggests that the effect of change in eigenfunction is accurately taken into account in that paper.

1. Introduction

Early studies of boundary layer stability used the 'parallel-flow' approximation, i.e. they neglected the effects of streamwise growth of the boundary layer on the grounds that, since this growth was slow (being $O(R^{-1})$ in the local boundary layer Reynolds number, R), its effects on stability were likely to be 'small'. Hence, the Orr–Sommerfeld equation, which is applicable to strictly parallel flows such as the fully developed flow in a channel, was used to obtain quantitative results for boundary layers also. The logical improvement of this approximation was to include the effects of spatial development, which has been done by several authors in the last two decades (e.g. Gaster 1974, hereafter G74; Smith 1979; Bertolotti, Herbert & Spalart 1992; Saric & Nayfeh 1975) using rather different approaches.

Gaster was the first to develop a method wherein all effects up to a given order in integral powers of R^{-1} could be consistently included. In this approach, non-parallel effects appear as higher-order corrections to the Orr–Sommerfeld operator. For nearly two decades after the publication of Gaster's work, there was a great deal of controversy about the subject, since different workers obtained different results from stability analyses, and there were no new experiments to compare with. Over time, however, these differences were all traced to inconsistencies in the methods employed, until all the different approaches gave stability results which essentially agreed with

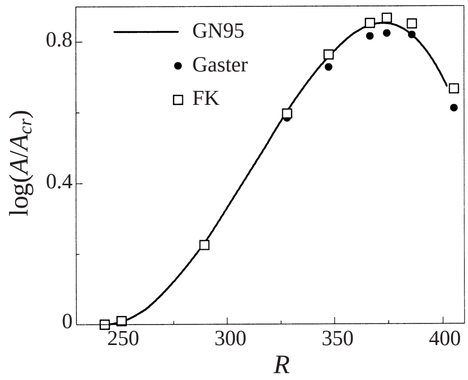

FIGURE 1. Amplitude ratio at the inner maximum in Blasius flow; $\omega/R = 1.4 \times 10^{-4}$. Symbols from Gaster (1974) and Fasel & Konzelmann (1990) (FK).

those of Gaster. His approach was further vindicated by direct numerical simulations (Fasel & Konzelmann 1990) and by the accurate experiments of Klingmann *et al.* (1993). The early non-parallel results of Gaster are therefore noteworthy.

The theory of Govindarajan & Narasimha (1995, referred to hereafter as GN95) follows a different approach: it begins from first principles and results in a stability equation for spatially developing boundary layers *nominally* correct to $O(R^{-1})$ (i.e. including all terms with the factor R^{-1} in the primitive equations). The approach and the solution method are described in § 3: unlike G74 it does not make any use of parallel-flow results. The final results obtained by G74 and by GN95, say for the stability loop, are so close that they have been considered to be in good agreement (see e.g. figure 2, GN95). However, when amplitude ratios are computed, a small discrepancy becomes noticeable: this is shown in figure 1, which is reproduced from figure 6 of GN95. It will be seen, however, that the GN95 theory is in excellent agreement with the full numerical simulations of Fasel & Konzelmann (1990) (see also figure 2). The present investigation has the aim of identifying, if possible, the reason for this discrepancy: although the discrepancy appears so small that it may not be considered of great practical value, it seemed worthwhile to see whether there was an issue of principle or method, as the magnitude of such a discrepancy could increase in e.g. adverse pressure gradient flows, which are more sensitive to approximations of this kind (see e.g. Govindarajan & Narasimha 1999, referred to hereafter as GN99). Furthermore, an acceptable explanation of the discrepancy should enhance confidence in stability computations, especially of the streamwise evolution of the amplitude ratios, which forms a crucial step in the estimation of transition location in technological applications (e.g. by the use of the e^n method). The chief objective of the present work is to focus attention on the order of magnitude of terms included or omitted in the different approaches, and hence to help arrive at self-consistent theories.

2. Gaster's formulation and solution method

We propose first to show that Gaster's formulation, if truncated at $O(R^{-1})$ and recast in the coordinate system of GN95, leads to a stability equation identical to theirs in the absence of pressure gradients. As mentioned before, Gaster considers the non-parallel solution to be a higher-order correction to the Orr–Sommerfeld solution.

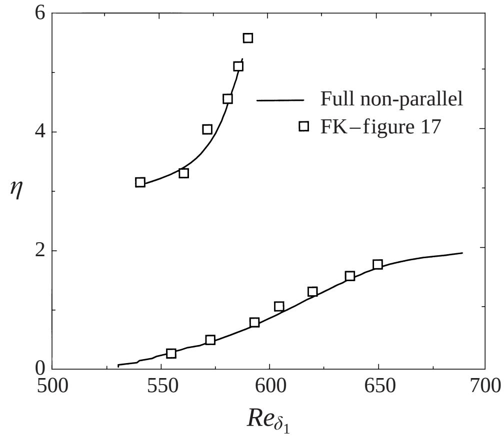

FIGURE 2. Dependence of critical Reynolds number on the normal coordinate: shown here in the units (based on displacement thickness, δ_1) used by Fasel & Konzelmann (1990).

In particular, the disturbance eigenfunction is taken to be the Orr–Sommerfeld eigenfunction ϕ_0 modulated by a spatially varying amplitude A with an additional small correction term:

$$\phi_G(\xi,\zeta) = A(\xi)\phi_0(\xi,\zeta) + \epsilon\phi_1(\xi,\zeta), \tag{2.1}$$

where $\xi = x_d/x_0$ is proportional to the dimensional value of the downstream distance, x_d, x_0 being a constant reference length. The normal variable ζ is proportional to y, as discussed in the Appendix. The correction term in (2.1), however, does not appear in the solution process described in G74; the important steps in the argument are reproduced below for clarity. The explicit non-parallel contribution can be lumped together in the Gaster operator $\{\mathrm{NP_G}\}$ and written down as

$$\{\mathrm{OS}\}\phi_G + \frac{1}{R_{x0}^{1/2}}\{\mathrm{NP_G}\}\phi_G = 0, \tag{2.2}$$

where $\{\mathrm{OS}\}$ is the Orr–Sommerfeld operator (defined in the present coordinates in equation (3.2) below), with the boundary conditions

$$\phi_G = \frac{\partial\phi_G}{\partial\zeta} = 0 \quad \text{at} \quad y = 0, \quad \text{and} \quad \phi_G \to 0, \frac{\partial\phi_G}{\partial\zeta} \to 0 \quad \text{as} \quad y \to \infty. \tag{2.3}$$

For clarity of discussion, equation (2.2) is rewritten, using the Reynolds number $R \equiv \theta U/\nu$ (where θ and U respectively are the momentum thickness and the free-stream velocity and ν is the kinematic viscosity), as

$$\{\mathrm{OS}\}\phi_G + \frac{(2q\xi)^{-1/2}}{R}\{\mathrm{NP_G}\}\phi_G = 0. \tag{2.4}$$

The Reynolds number used in G74 is related to the present definition by

$$R_{x0}^{1/2} = \frac{R}{\sqrt{2q\xi}}. \tag{2.5}$$

Since

$$\{\mathrm{OS}\}\phi_0 = 0 \tag{2.6}$$

by definition, equation (2.4) may be simplified, using (2.1), to

$$\left(\frac{1}{R}\right)\{\mathrm{NP_G}\}[A\phi_0] = -\epsilon(2q\xi)^{-1/2}\{\mathrm{OS}\}\phi_1 - \epsilon\left(\frac{1}{R}\right)\{\mathrm{NP_G}\}[\phi_1]. \tag{2.7}$$

From the above equation it seems reasonable to take $\epsilon = R^{-1}$. If ϵ is prescribed to be of this order, the second term on the right-hand side appears at first sight to be of higher order, and has on these grounds been dropped by Gaster. This neglect of $\{\mathrm{NP_G}\}[\phi_1]$ turns out to be the primary cause for the difference between Gaster's and GN95's approaches, and will be discussed in detail in §4.

Setting $\epsilon = R^{-1}$ and dropping the last term, equation (2.7) is rewritten by Gaster as

$$\{\mathrm{NP_G}\}[A\phi_0] = -(2q\xi)^{-1/2}\{\mathrm{OS}\}\phi_1, \tag{2.8}$$

or (collecting terms involving A),

$$AF_0 + \frac{\mathrm{d}A}{\mathrm{d}\xi}F_1 + (\text{higher-order terms}) = (2q\xi)^{-1/2}\{\mathrm{OS}\}\phi_1, \tag{2.9}$$

where F_0 and F_1 are functions involving only ϕ_0.

Following a procedure similar to that used by Stuart (1960), we can write

$$\int_0^\infty \chi\{\mathrm{OS}\}[\rho]\,\mathrm{d}y = \int_0^\infty \rho\overline{\{\mathrm{OS}\}}\,[\chi]\,\mathrm{d}y, \tag{2.10}$$

where $\rho(y)$ is any function with homogeneous boundary conditions at the wall and in the free-stream on itself and its first derivative (such as in equation (2.3)), χ is the adjoint eigenfunction to the Orr–Sommerfeld equation, and $\overline{\{\mathrm{OS}\}}$ is the corresponding adjoint operator. The function ρ need not satisfy the Orr–Sommerfeld equation. By definition, the right-hand side of (2.10) is identically zero and therefore

$$\int_0^\infty \chi\{\mathrm{OS}\}[\rho]\,\mathrm{d}y = 0. \tag{2.11}$$

The eigenfunctions ϕ_G and ϕ_0 both satisfy the boundary conditions in (2.3). Hence, from equation (2.1) it can be seen that ϕ_1 also satisfies the same boundary conditions. Therefore equation (2.11) may be applied to ϕ_1:

$$\int_0^\infty \chi\{\mathrm{OS}\}[\phi_1]\,\mathrm{d}y = 0. \tag{2.12}$$

Dropping the higher-order terms in (2.8), the amplitude function A is then given by the ordinary differential equation

$$A(\xi)\int_0^\infty F_0\chi\,\mathrm{d}y + \frac{\mathrm{d}A}{\mathrm{d}\xi}\int_0^\infty F_1\chi\,\mathrm{d}y = 0. \tag{2.13}$$

Since F_0 and F_1 are known, the streamwise variation of amplitude can be obtained from (2.13), without having to solve for the non-parallel contribution ϕ_1 to the eigenfunction.

3. The approach of GN95

In zero pressure gradient, the non-parallel stability equation of GN95 can be written

$$\{\mathrm{OS}\}\phi + \frac{1}{R}\{\mathrm{NP}\}\phi = 0, \tag{3.1}$$

with boundary conditions on ϕ as in (2.3). The operators $\{\mathrm{OS}\}$ and $\{\mathrm{NP}\}$ are respectively

$$\{\mathrm{OS}\} \equiv \mathrm{i}(\omega - \alpha\Phi')(\mathrm{D}^2 - \alpha^2) + \mathrm{i}\alpha\Phi''' + \frac{1}{R}(\mathrm{D}^4 - 2\alpha^2\mathrm{D}^2 + \alpha^4), \tag{3.2}$$

$$\{\mathrm{NP}\} \equiv q(\Phi\mathrm{D}^3 + \Phi'\mathrm{D}^2 + [2y\alpha(\omega - \alpha\Phi') - \alpha^2\Phi + \Phi'']\,\mathrm{D} + [-\alpha\omega + \Phi'''])$$
$$+(-\omega + 3\alpha\Phi')R\alpha' + [\Phi''' + 3\alpha^2\Phi' - 2\alpha\omega - \Phi'\mathrm{D}^2]R\frac{\partial}{\partial x}. \tag{3.3}$$

Here θ and U have been used as scales. The operator D stands for differentiation with respect to the normal coordinate y, y being proportional to the Blasius similarity variable, and the streamwise coordinate x is non-dimensionalized in a special way:

$$y_d = \theta y \quad \text{and} \quad \mathrm{d}x_d = \theta\,\mathrm{d}x. \tag{3.4}$$

Φ and ϕ are respectively the mean and the disturbance amplitude of the streamfunction ψ:

$$\psi(x, y, t) = \Phi(y) + \phi(x, y)\exp\left[\mathrm{i}\left(\int \alpha\,\mathrm{d}x - \omega t\right)\right]. \tag{3.5}$$

For the flow over a flat plate, the parameter q, defined by the relation

$$\frac{q}{R} \equiv \frac{\mathrm{d}\theta}{\mathrm{d}x}, \tag{3.6}$$

is constant. Equation (3.1) is derived from the two-dimensional incompressible Navier–Stokes equation in streamfunction form assuming linear normal mode disturbances; details are available in GN95. All terms with a factor R^{-1} are retained and higher-order terms are neglected. The analysis assumes that $\partial\phi/\partial x$ and α' are $O(R^{-1})$, and that the second and higher derivatives with respect to x are of higher order.

Consider the following representation of equation (3.1):

$$\{\mathrm{D}^4 + b_3\mathrm{D}^3 + b_2\mathrm{D}^2 + b_1\mathrm{D} + b_0\}\phi = B_x\frac{\partial\phi}{\partial x}, \tag{3.7}$$

where b_0 to b_3 are functions of y and x (the latter through the Reynolds number, wavenumber etc.), and B_x is an operator in y. Since $\partial\phi/\partial x$ is small, its variation in the downstream direction is negligible, so the right-hand side of this equation is primarily a function of y. Hence the partial differential equation (3.7) can be solved iteratively, first solving the equation putting the right-hand side of (3.7) equal to zero, then estimating the value of $\partial\phi/\partial x$, etc., as described in detail in GN95.

4. Discussion

The expression for the ‘full’ non-parallel operator is not written out explicitly in G74, but it can be indirectly deduced using the expressions for F_0 and F_1 as given in Appendix A of that paper. On transforming this expression to the coordinate system of GN95, it is found that Gaster’s non-parallel operator is the same as that given by GN95 in their equation (5), as shown in the Appendix. Thus, in the case of flow over a flat plate, G74 and GN95 are equivalent formulations, both correct to $O(R^{-1})$. It therefore emerges that the primary cause for the quantitative difference in the results from the two approaches, seen in figure 1, must be that the term $(\epsilon/R)\{\mathrm{NP_G}\}\phi_1$ in equation (2.7) is neglected by Gaster but included in the analysis of GN95. The argument below shows that while neglecting the term is entirely justifiable in the bulk of the boundary layer, the term is of an order lower than or equal to R^{-1} in the critical and wall layers and must therefore be retained in an analysis correct to that order. By neglecting this term, the correction ϕ_1 to the shape of the Orr–Sommerfeld eigenfunction need not be evaluated at all, and Gaster achieves a considerable operational advantage. The properties of the Orr–Sommerfeld operator and its adjoint eigenfunction can be used to obtain non-parallel effects as a perturbation on the 'locally parallel' result – but at a cost to be determined below. GN95, on the other hand, do not consider the non-parallel eigenfunction to be a correction on the Orr–Sommerfeld eigenfunction: they directly obtain the non-parallel eigenfunction ϕ and do not neglect any part of it.

In the bulk of the boundary layer (i.e. the region which is not within either the critical or the wall layer), the factor ϵ in equation (2.7) may be set equal to R^{-1}, and the second term on the right-hand side may indeed be neglected. However, as discussed below, the neglect of this term is not valid in the critical and wall layers. In these layers, the disturbance eigenfunction ϕ is expressed in the form of asymptotic expansions respectively as (GN99)

$$\phi(y) = \sum_k \epsilon_c^k \gamma_{ck}(\eta_c) + \sum_m \epsilon_c^m (\log \epsilon_c) \lambda_{cm}(\eta_c), \tag{4.1}$$

and

$$\phi(y) = \sum_k \epsilon_w^k \gamma_{wk}(\eta_w) + \sum_m \epsilon_w^m (\log \epsilon_w) \lambda_{wm}(\eta_w), \tag{4.2}$$

$$k = 0, 1, 2, \ldots, \quad m = 1, 2, \ldots .$$

Here $\epsilon_c \sim R^{-1/3}$ and $\epsilon_w \sim R^{-1/2}$ as shown in GN99. The relevant normal variables in the critical and wall layers, respectively, are

$$\eta_c = \frac{(y - y_c)}{\epsilon_c} \quad \text{and} \quad \eta_w = \frac{y}{\epsilon_w}, \tag{4.3}$$

y_c being the critical height (at which the mean flow velocity equals the phase velocity of the disturbance).

In the critical layer, therefore, differentiation with respect to the relevant normal variable lowers the order by $R^{1/3}$ (Drazin & Reid 1981; Govindarajan & Narasimha 1997, hereafter GN97; GN99). Returning to equation (2.7), we can see that the second term on the right-hand side is only $O(R^{-1/3})$ compared to the left-hand side. Since $\{\mathrm{NP_G}\}$ contains a third-derivative term, it can be shown that

$$\frac{\epsilon_c}{R}\{\mathrm{NP_G}\}[\phi_1] = O(R^{-2/3}) \tag{4.4}$$

relative to the largest terms in the full non-parallel equation (2.4) and is thus not negligible in the critical layer.

In the wall layer, setting $\epsilon_w = (\alpha R)^{-1/2}$ makes

$$\frac{\epsilon_w}{R}\{\mathrm{NP_G}\}[\phi_1] = O(R^{-1}) \tag{4.5}$$

relative to the largest terms in the wall layer. In a rational theory correct to and including $O(R^{-1})$, this term may not be neglected.

The resulting difference is 'small', i.e. we expect it to make little numerical difference in eigenvalue computations. This is indeed the case, as shown in GN95, where the neutral stability boundaries agree closely with the results of Gaster. However, in the computations of downstream growth of disturbance amplitude, the effect of these discrepancies is cumulative. Hence the small but noticeable differences in the two results, as demonstrated in figure 1.

Before we end, it is necessary to note that the present conclusions are not affected by the recent work of Gaster (2000), who has proposed an approximate solution method for the full non-parallel equation by the appropriate scaling of a related parallel-flow solution. The reason is that it cannot yield an amplitude distribution that is of greater accuracy than G74. To relate the present work to Gaster's, it is important to note that, while the theories for non-parallel flows formulated in GN95, GN97 and GN99 all adopt the same philosophy and mathematical approach – namely that of formulating what we have called minimal composite equations, as explained in detail in Narasimha & Govindarajan (2000) – the equation derived in each of the three papers is different from the others, as they reflect different levels of approximation in the minimal composite theory. More specifically, GN95, GN97, GN99 may be thought of as correct respectively to $O(R^{-1})$, $O(R^{-1/2})$ and $O(R^{-2/3})$ in the limit as the local Reynolds number $R \to \infty$.

The outcome of GN99 is a 'lowest order parabolic' partial differential equation, in which a term involving a derivative of the eigenfunction in the streamwise coordinate appears explicitly in the equation. The eigenfunction in GN99 is therefore completely specified, and the comments to the contrary in Gaster (2000) clearly cannot apply. Similarly, the eigenfunctions are also specified completely in GN95, which formulates a partial differential equation more accurate than GN99.

Naturally the eigenfunction cannot be completely specified in the above sense whenever the stability of a spatially developing shear flow is to be treated by an ODE. This is true for the solutions of the equation formulated in GN97, as also of all work done with the Orr–Sommerfeld equation. Indeed, it was for this reason that GN99 (see p. 1450) sought to formulate, using again minimal composite theory, the lowest-order equation that explicitly enabled the complete specification of the eigenfunction. The chief contribution of GN97, on the other hand, is the demonstration that there is a rational non-parallel flow ODE, different from Orr–Sommerfeld, which includes explicitly the effect of non-parallelism; thus there is one term in the GN97 equation that is directly proportional to the rate of growth of a boundary layer thickness, and appears because of the mean normal velocity which is necessarily absent in a parallel flow/Orr–Sommerfeld type of theory. However, the problem of specification of the dependence on x remains with GN97 (and the Orr–Sommerfeld equation), and has usually been and continues to be handled through engineering approximations in which the equation is solved locally at each streamwise station and the solutions at neighbouring stations are patched using an (unstated) principle of the streamwise continuity of the amplitude distribution. This problem can be tackled in different ways, but it is not our purpose here to describe the logic of these procedures.

The iterative scheme of GN95 (contrary to the suggestion in Gaster 2000) is in fact robust and has been used to compute amplification factors over a variety of aerofoil sections. The scheme, therefore, and the philosophy underlying it, seem satisfactory.

5. Conclusions

It has been shown here that, while the equation of GN95 and the parabolized stability equation of Bertolotti *et al.* (1992) are equivalent to that of G74, minor inconsistencies do arise from the methods of solution adopted when (as in G74) the Orr–Sommerfeld eigenfunction is used to compute the streamwise evolution of the amplitude. This approach permits a useful exploitation of the properties of adjoint operators, but the results so obtained are correct only to $O(R^{-1/2})$, whereas the governing equations are more accurate (to $O(R^{-1})$). The computations of GN95, exploiting as they do the full accuracy of the governing equations, eliminate the small discrepancy, and are in excellent agreement with direct numerical simulation results. These results enhance confidence in the approach adopted in GN95, GN97, GN99.

This work is supported by the Naval Research Board, Government of India. We are grateful to Professor M. Gaster for several discussions, and for his comments on an early version of this manuscript.

Appendix

It is shown below that, up to the order considered, the operators $\{\mathrm{NP}\}$ in equation (3.1) and $\{\mathrm{NP_G}\}$ in equation (2.4) are equivalent. The left-hand side of (2.9) may be expanded using the expressions for F_0 and F_1 (up to $O(R^{-1})$) from equation (A6) of G74:

$$\begin{aligned}\{\mathrm{NP_G}\}[A\phi_0] &= AF_0 + \frac{\mathrm{d}A}{\mathrm{d}\xi}F_1 \\ &= -A[2\alpha_G\omega_G - 3\alpha_G^2 f_\zeta - f_{\zeta\zeta\zeta}]\left[\frac{\partial\phi_0}{\partial\xi} - \frac{\zeta}{2\xi}\phi_{0\zeta}\right] \\ &\quad -Af_\zeta\left\{\frac{\partial\phi_{0\zeta\zeta}}{\partial\xi} - \frac{\zeta}{2\xi}\phi_{0\zeta\zeta\zeta} - \frac{\phi_{0\zeta\zeta}}{\xi}\right\} - (\omega_G - 3\alpha_G f_\zeta)\left(\frac{\mathrm{d}\alpha_G}{\mathrm{d}\xi} - \frac{\alpha_G}{2\xi}\right)A\phi_0 \\ &\quad +\frac{A}{2\xi}\{(f_{\zeta\zeta} + \zeta f_{\zeta\zeta\zeta})\phi_{0\zeta} + (f - \zeta f_\zeta)(\phi_{0\zeta\zeta\zeta} - \alpha_G^2\phi_{0\zeta})\} \\ &\quad -\frac{\mathrm{d}A}{\mathrm{d}\xi}\{[2\alpha_G\omega_G - 3\alpha_G^2 f_\zeta - f_{\zeta\zeta\zeta}]\phi_0 + f_\zeta\phi_{0\zeta\zeta}\}. \qquad \text{(A 1)}\end{aligned}$$

The Gaster eigenfunction $\phi_G = U\theta\phi$. For the flow over a flat plate, U is constant, and

$$\phi_G \propto \xi^{1/2}\phi. \qquad \text{(A 2)}$$

In order to rephrase Gaster's non-parallel operator in such a way as to compare directly with equation (3.3), we make use of (A 2) to define a new variable

$$\upsilon = \frac{A\phi_0}{\xi^{1/2}}, \qquad \text{(A 3)}$$

in terms of which equation (A 1) can be written as

$$\begin{aligned}\{\mathrm{NP_G}\}[A\phi_0] = -\xi^{1/2}\Bigg\{&[2\alpha_G\omega_G - 3\alpha_G^2 f_\zeta - f_{\zeta\zeta\zeta}]\left[\frac{\partial\upsilon}{\partial\xi} + \frac{\upsilon}{2\xi} - \frac{\zeta}{2\xi}\upsilon_\zeta\right] \\ &+f_\zeta\left[\frac{\partial\upsilon_{\zeta\zeta}}{\partial\xi} - \frac{\zeta}{2\xi}\upsilon_{\zeta\zeta\zeta} - \frac{\upsilon_{\zeta\zeta}}{2\xi}\right] + (\omega_G - 3\alpha_G f_\zeta)\left(\frac{\mathrm{d}\alpha_G}{\mathrm{d}\xi} - \frac{\alpha_G}{2\xi}\right)\upsilon \\ &+(f_{\zeta\zeta} + \zeta f_{\zeta\zeta\zeta})\upsilon_\zeta + (f - \zeta f_\zeta)(\upsilon_{\zeta\zeta\zeta} - \alpha_G^2\upsilon_\zeta)\Bigg\}. \qquad \text{(A 4)}\end{aligned}$$

Here ζ is Gaster's normal variable given by

$$\zeta = \frac{U}{\nu}\frac{y_d}{(R_{x0}\xi)^{1/2}} = \sqrt{2q}\,y, \tag{A 5}$$

$f_\zeta = \Phi'$, α_G and ω_G respectively are the wavenumber and frequency of the disturbance in Gaster's coordinate system. From equation (6) of G74 it is straightforward to obtain

$$\alpha_G = \frac{\alpha}{\sqrt{2q}}, \tag{A 6}$$

and it may be inferred that $\omega_G = \omega/\sqrt{2q}$. The relations for differentiation with respect to dimensional variables in the two coordinate systems can be equated, giving

$$\frac{U}{\nu R_{x0}}\left\{\frac{\partial}{\partial\xi} - \frac{\zeta}{2\xi}\frac{\partial}{\partial\zeta}\right\} = \frac{1}{\theta}\left\{\frac{\partial}{\partial x} - \frac{yq}{R}\frac{\partial}{\partial y}\right\} \tag{A 7}$$

and

$$\frac{U}{\nu(\xi R_{x0})^{1/2}}\frac{\partial}{\partial\zeta} = \frac{1}{\theta}\frac{\partial}{\partial y}. \tag{A 8}$$

Equations (A 5) to (A 8) may be used to rewrite (A 4) as

$$\begin{aligned}4q\sqrt{\xi}\{\mathrm{NP_G}\}[A\phi_0] = &-[2\alpha\omega - 3\alpha^2\Phi' - \Phi'''] \left[\frac{R}{q}\frac{\partial \upsilon}{\partial x} + \upsilon - y\mathrm{D}\upsilon\right] \\ &-\Phi'\left[\frac{R}{q}\frac{\partial \mathrm{D}^2\upsilon}{\partial x} - y\mathrm{D}^3\upsilon - \mathrm{D}^2\upsilon\right] - (\omega - 3\alpha\Phi')\left(\frac{R}{q}\frac{\mathrm{d}\alpha}{\mathrm{d}x} - \alpha\right)\upsilon \\ &+(\Phi'' + y\Phi''')\mathrm{D}\upsilon - (y\Phi' - \Phi)(\mathrm{D}^3\upsilon - \alpha^2\mathrm{D}\upsilon).\end{aligned} \tag{A 9}$$

Combining terms in υ, $\mathrm{D}\upsilon$ etc. we get

$$\begin{aligned}4q^2\sqrt{\xi}\{\mathrm{NP_G}\}[A\phi_0] = q\,&\left\{\Phi\mathrm{D}^3 + \Phi'\mathrm{D}^2 + \left[2y\alpha(\omega - \alpha\Phi') - \alpha^2\Phi + \Phi''\right]\mathrm{D}\right. \\ &\left.+(\Phi''' - \alpha\omega)\right\}\upsilon + (3\alpha\Phi' - \omega)R\frac{\mathrm{d}\alpha}{\mathrm{d}x}\upsilon \\ &+R\left[-2\alpha\omega + 3\alpha^2\Phi' + \Phi''' - \mathrm{D}^2\right]\frac{\partial\upsilon}{\partial x}.\end{aligned} \tag{A 10}$$

On examining the inviscid operators of the Orr–Sommerfeld equation on the left-hand sides of (3.1) and (2.4) it is seen that Gaster's inviscid operator is $(2q)^{3/2}$ times the corresponding operator in (3.1). When this is accounted for, it is seen that the Gaster non-parallel operator (appearing in (2.4)) is the same as that of GN95 (equation (3.3)). The only difference (as discussed in the main text) is that the former acts only on υ while the latter acts on the total eigenfunction ϕ.

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